The complete generality of Euler's rotations nonetheless assures that there must be values of ψ, θ, φ which correspond to all rotations, including arbitrary infinitesimal rotations about three orthogonal directions. These are found by considering those sets of three Euler rotations which leave the triad invariant, the final configuration congruent to the initial. One of these is $\psi = \pi/2$, $\theta = \pi/2$, $\varphi = 0$, leading to the required infinitesimal rotation by means of the angles

$$\psi = (\pi/2) + \alpha \qquad \theta = (\pi/2) + \gamma \qquad \varphi = \beta \quad (3)$$

When the angles α, β, γ are small,

$$\alpha \ll 1 \qquad \beta \ll 1 \qquad \gamma \ll 1 \quad (4)$$

the substitution of (3) into (1) recovers (2). The suitability of angles (3) is also geometrically evident because the two rotations of approximate magnitude $\pi/2$ have the effect of separating each rotation axis from the others by an angular distance $\pi/2$. This is clearly necessary for the representation of a resultant small rotation of arbitrary direction, but it is precluded by the Euler sequence when ψ or θ is limited to small values, the remaining angle φ also being small.

The usefulness of a set of rotations such as (3) (other sets may also be found which serve equally well) is that accurate calculation of libration amplitudes, mode coupling, stability, and related characteristics requires consideration of both infinitesimal and finite values of α, β, γ in unrestricted combinations; angles (3) permit the partially stabilized libration to be joined to the completely stabilized libration in which $\alpha = \beta = \gamma = 0$. This is accomplished, moreover, within the orthodox framework of Euler rotations in a generality not possible with the small values of ψ, θ, φ, which correctly represent only the limiting state of zero libration.

References

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Stationary Earth Orbits

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As observed from the earth, the orbit of an earth satellite is a Keplerian ellipse on which is superposed a rotation equal in magnitude but opposite in sign to that of the earth. It is possible to remove this precession by a suitably programed continuous thrust, leading to an orbit that is "stationary" with respect to the earth.† The simplest example is a circular polar orbit constrained to remain in a plane of fixed longitude. Such orbits would have obvious advantages for a system of communication satellites.

Although the maintenance for long times of orbits that are stationary in the forementioned sense imposes requirements that are far beyond what is at present attainable, it seems worthwhile to sketch the very simple theory that establishes qualitatively what these requirements are.

Assuming that a reference frame in which the earth is at rest is an inertial frame, the equation of motion of a point satellite in a frame of reference rotating with the earth takes the well-known form

$$d^2\mathbf{r}/dt^2 = \mathbf{a}(\mathbf{r}) + \mathbf{S} - 2\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{a}$ is the earth's gravitational acceleration at a point $\mathbf{r}$, $\mathbf{S}$ is the acceleration produced by the programed thrust, $\boldsymbol{\omega}$ the angular velocity of rotation of the earth, and $\mathbf{v}$ the velocity of the satellite relative to the earth. The last two terms are, respectively, the Coriolis and centrifugal accelerations introduced by the transformation to a coordinate system with rotation $\boldsymbol{\omega}$. A Keplerian ellipse stationary with reference to the earth is obtained by programing the thrust to give the acceleration‡:

$$\mathbf{S} = 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

It is readily shown that for low earth orbits the Coriolis term $2\boldsymbol{\omega} \times \mathbf{v}$ is predominant (more than 90% of the total acceleration out to five earth radii), and the order of magnitude of the average specific thrust $G = S/g$, where g is the acceleration of gravity at the surface of the earth, is given by $G = \omega v/g$. For $v = 26{,}000$ fps, that is, for orbital speed at the surface of the earth, $G = 0.06$. Although this is a relatively low specific thrust, the powerplant or propellant requirements for maintenance of a stationary orbit for long periods of time are prohibitively high. What these requirements are may be seen by considering the simple but typical case of a circular polar orbit at low altitude. The average over 1 rev of $|2\boldsymbol{\omega} \times \mathbf{v}|$ is then nearly equal to ωv, and the equation for propellant consumption may be written as $Ig(dm/dt) = -m\omega v$, where I is the specific impulse and v may be taken as constant. Integration of this equation leads immediately to $\ln(m_0/m_f) = (\omega v/Ig)\Delta t$, where Δt is the time in which the satellite mass is reduced from m_0 to m_f by propellant consumption. For an idealized situation in which the initial mass m_0 is composed of propellant, payload (mass m_L), and powerplant plus propulsion equipment (mass m_E), the equation foregoing for Δt can be put in the form

$$\Delta t = -(Ig/\omega v) \ln[l + (m_E/m_0)] \quad (1)$$

where $l = m_L/m_0$ is the payload ratio.

For the chemical propulsion, m_E/m_0 may be neglected to a first approximation if l is not too small. Equation (1) then becomes, for $I = 400$ sec and $G = \omega v/g = 0.06$ as for low earth orbit, $\Delta t = 6800 \ln l$ sec. For $l = 0.5$, Δt is of the order of the time of 1 rev of the satellite. If there should be a requirement for such a polar orbit that returns to the same point on the earth it could, however, be met more easily by a single impulsive change in velocity than by a continuous programed thrust.

Since Δt in Eq. (1) increases only logarithmically as l decreases but is proportional to I, longer times can best be attained by increasing the specific impulse. This suggests electrical propulsion, in which I can be made very large and programing of the thrust is relatively easy. However, the power required and accordingly the mass of the powerplant will also increase with I, and a maximum Δt for an optimum I is to be expected. That this is the case when the powerplant mass is proportional to the maximum jet power is easily seen analytically. Writing $m_E = bP$, where b is a constant and P the maximum power, leads at once to the relation $m_E/m_0 = \frac{1}{2}g^2bIG$. Inserting this in Eq. (1) and using the notation $x = l + \frac{1}{2}g^2bIG$ shows that $\Delta t \sim (x - l) \ln x$. The value x_0 of x which maximizes Δt for fixed l, b, and G can then be obtained by setting $(d/dx)(x - l) \ln x_0 = 0$. This results in the implicit equation $\ln x_0 = (l - x_0)/x_0$ for the determination of x_0. The maximized time Δt_m is then given by

$$\Delta t_m = \frac{2(x_0 - l)}{g^2G^2b} \ln\frac{1}{x_0} = \frac{5.8 \times 10^3(x_0 - l)}{b} \ln\frac{1}{x_0} \text{ sec} \quad (2)$$

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† It is the orbit that is stationary, not the position of the satellite as in the very special case of a 24-hr equatorial circular orbit.

‡ There is an amusing analogy to the Zeeman effect. The orbit of an atomic electron in a magnetic field $\mathbf{H}$ is stationary in a coordinate system having the Larmor precession $\boldsymbol{\omega}_L$ given by $(e/mc)\mathbf{v} \times \mathbf{H} = 2\boldsymbol{\omega}_L \times \mathbf{v}$, that is, one in which acceleration by the Lorentz force cancels the Coriolis acceleration.

and the optimum specific impulse I_0 by

$$I_0 = \frac{2(x_0 - l)}{g^2 G b} = \frac{350(x_0 - l)}{b} \text{ sec} \qquad (3)$$

where the second equalities are for b expressed in kilograms per kilowatt and for $G = 0.06$, as for a low-altitude polar orbit. It is also easy to show that the power level is given by

$$P = 2.9 \times 10^{-3} I_0 m_0 \text{ kw} \qquad (4)$$

where $m_0 = m_L/l$ is the initial total mass in kilograms. Typical values of x_0 obtained graphically are $l = 0.1$, $x_0 = 0.46$, $l = 0.5$, $x_0 = 0.72$.

For a payload ratio $l = 0.1$, Eq. (2) may be written $\Delta t_m = (10^{-2}/b)$ days. The corresponding optimum specific impulse from Eq. (3) is $I_0 \approx (125/b)$ sec. It is evident that a specific powerplant mass of about 10^{-2} kg/kw and a specific impulse of about 10^4 sec are required for a duration of even one day. Such a specific powerplant mass is at least two orders of magnitude lower than present design objectives, and one day is too short a time to be of interest for communication satellites, although it might be acceptable for certain special applications. Lower values of $G = \omega v/g$ and, hence, increased duration can be obtained by going to larger circular orbits. However, as v decreases only as the square root of the ratio of the earth's radius to the radius of the orbit, no more than a factor of about 2 can be gained in this way without approaching the 24-hr orbit at six earth radii.

There is no solid basis for speculation as to whether the required specific powerplant masses might become feasible in the future. However, it is worth noting that for sizable payloads the power level given by Eq. (4) is quite high. For example, for a 200 kg payload it approaches 100 mw for $l = 0.1$. In such a power range, radiator mass is predominant, and progress may be largely dependent on radical advances in the field of heat rejection.

Some Energy and Momentum Considerations in the Perforation of Plates

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An early interpretation of experimental data on armor perforation by small metal fragments, moving at ordnance speeds, was made by considering momentum to be conserved between the impacting fragment and the target mass thrown out, leading to the relation

$$v_r/v_s = 1/(1 + \alpha) \qquad (1)$$

where v_r is the fragment residual velocity, v_s the fragment striking velocity, and α the ratio of the target mass thrown out to the fragment mass. The residual velocity of the fragment is assumed to be equal to the ejection velocity of the target pieces. There is good agreement with steel target thicknesses up to about 0.5 cm between this simple expression and the data obtained by Spells.[1] Furthermore, the formula predicts fairly well the data obtained by Jameson and Williams.[2] In general, the experimental data are predicted more closely for small target thicknesses and when v_s is well above the limiting velocity for perforation.

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A similarly elementary but somewhat more rigorous treatment is possible by applying the conservation of energy and momentum to the complete system. We consider the perforation process as a collision between the fragment, the ejected target pieces, and the bulk of the target. On the basis of this model, the following assumptions are made. The energy lost to the target is a constant for a given target. The transfer of momentum to the target pieces is by direct inelastic impact. Impact occurs perpendicular to the surface and at the center of the mass of the target. Furthermore, the pieces emerge with approximately the same velocity, namely, v_r, which is in good agreement with the data, and the momenta of the target pieces are parallel to that of the fragment. Then, the energy and momentum equations, respectively, take the form

$$\tfrac{1}{2}mv_s^2 = \tfrac{1}{2}(m + m_t)v_r^2 + \tfrac{1}{2}(M - m_t)V^2 + W \qquad (2)$$

$$mv_s = (m + m_t)v_r + (M - m_t)V \qquad (3)$$

where m and m_t are, respectively, the masses of the fragment and target pieces, M and V are the mass and velocity of the target, and W is what may be called the work of penetration, which includes shock-wave energy dissipation. In general, W will depend on several variables, including the impact velocity, the properties of the target material, and the properties and shape of the fragment.

Eliminating V between Eqs. (2) and (3) and solving for the ratio v_r/v_s results in

$$\frac{v_r}{v_s} = a + \left[a^2 + \frac{a}{bc}\left(1 - b - \frac{2W}{mv_s^2}\right)\right]^{1/2} \qquad (4)$$

The positive root is chosen since v_r/v_s must be positive on physical grounds. The symbols are defined by

$$a = \mu/(1 + \mu) \qquad b = \mu/(1 + \lambda) \qquad c = 1 + \alpha$$

where $\mu = m/M$ and $\lambda = m_t/M$ are mass ratios. If the target mass M is appreciably larger than m and m_t, then $a \simeq \mu$ and $a/(bc) \simeq (1 - \lambda)/(1 + \alpha)$. Thus, (4) reduces to

$$\frac{v_r}{v_s} = \mu + \left[\mu^2 - \mu + \frac{1 - \lambda}{1 + \alpha}\left(1 - \frac{2W}{mv_s^2}\right)\right]^{1/2}$$

Now, if M is much larger than m and m_t in such a way that $M \to \infty$, then a second approximation, by inspection, results in

$$\frac{v_r}{v_s} = \left[\frac{1}{1 + \alpha}\left(1 - \frac{2W}{mv_s^2}\right)\right]^{1/2} \qquad (5)$$

The latter result is the expression one obtains from the energy equation (2) if the kinetic energy imparted to the target is neglected.

We now separate the inelastic energy loss from W by writing $W = E + E_0$, where E is the energy lost by inelastic impact between the fragment and the target pieces, and E_0 is the energy lost to the rest of the target. If E were the only energy loss, the energy and momentum equations could be written as

$$\tfrac{1}{2}mv_s^2 = \tfrac{1}{2}(m + m_t)\bar{v}_r^2 + E_T + E$$

$$mv_s = (m + m_t)\bar{v}_r + P_T$$

where the kinetic energy and momentum imparted to the target are now denoted by E_T and P_T, respectively. Eliminating $\bar{v}_r$ and solving for E gives, after rearranging,

$$E = \frac{1}{2}\left(\frac{\alpha}{1 + \alpha}\right) mv_s^2 \times \left\{\frac{1 + \alpha}{\alpha}\left(1 - \frac{2E_T}{mv_s^2}\right) - \frac{1}{\alpha}\left(1 - \frac{P_T}{mv_s}\right)^2\right\} \qquad (6)$$

This expression for E involves the quantities $2E_T/(mv_s^2)$ and $P_T/(mv_s)$, the former being the ratio of the kinetic energy of